

$$N = 231 = 3 \cdot 7 \cdot 11$$

By the Ibukiyama-Kitayama dimension formula,  
 $\dim(S_4(K(231))) = 106$

By the Skoruppa-Zagier dimension formula and Jacobi restriction,  
the lift dimension of  $S_4(K(231))^+$  is 36  
the nonlift dimension of  $S_4(K(231))^+$  is heuristically 59  
 $\dim(S_4(K(231))^+)$  thus is heuristically 95  
 $\dim(S_4(K(231))^-)$  is heuristically 11

The heuristic dimensions are correct by the spanning results to follow

$\dim(J_{\{2,231\}}^{\{\text{cusp}\}}) = 1$  (Skoruppa-Zagier), so need to span completely

$q = 5$  for TraceDown

After TD( $\text{Grit}(J_{\{4,1155\}}^{\{\text{cusp}\}})$ ) and  $(\text{Grit}(J_{\{2,231\}}^{\{\text{cusp}\}}))^2$ ,  
spanned rank in  $S_4(K(231))^+$  is 95  
spanned rank in  $S_4(K(231))^-$  is 0

After Borcherds products,  
spanned rank in  $S_4(K(231))^-$  is 11

Final spanned rank in  $S_4(K(231))^+$  is 95

Final spanned rank in  $S_4(K(231))^-$  is 11

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 $S_2(K(231))^+$  is determined by Jacobi restriction and the  $H4Nd1(1,+)$  test  
( $\dim(H_4(231,1,1))^+ \leq 1$  and this is less than  $\dim(J_{\{2,231\}}^{\{\text{cusp}\}})+1 = 2$ )  
 $S_2(K(231))^- = 0$  by Jacobi restriction and the  $H4Nd1(1,-)$  test  
( $\dim(H_4(231,1,1))^- \leq 0$  and this is less than  $\dim(J_{\{2,231\}}^{\{\text{cusp}\}}) = 1$ )

So  $S_2(K(231)) = \text{Grit}(J_{\{2,231\}}^{\{\text{cusp}\}})$  (dimension 1)