

$$N = 165 = 3 \cdot 5 \cdot 11$$

By the Ibukiyama-Kitayama dimension formula,  
 $\dim(S_4(K(165))) = 59$

By the Skoruppa-Zagier dimension formula and Jacobi restriction,  
the lift dimension of  $S_4(K(165))^+$  is 26  
the nonlift dimension of  $S_4(K(165))^+$  is heuristically 29  
 $\dim(S_4(K(165))^+)$  thus is heuristically 55  
 $\dim(S_4(K(165))^-)$  is heuristically 4

The heuristic dimensions are correct by the spanning results to follow

$\dim(J_{\{2,165\}}^{\{\text{cusp}\}}) = 2$  (Skoruppa-Zagier), so need to span to within 1 dimension

$q = 7$  for TraceDown

After TD( $\text{Grit}(J_{\{4,1155\}}^{\{\text{cusp}\}})$ ) and  $(\text{Grit}(J_{\{2,165\}}^{\{\text{cusp}\}}))^2$ ,  
spanned rank in  $S_4(K(165))^+$  is 55  
spanned rank in  $S_4(K(165))^-$  is 0

Hecke operators applied:  $\{\{3, 2\}\}$

After Hecke spreading,  
spanned rank in  $S_4(K(165))^-$  is 2

After Borcherds products,  
spanned rank in  $S_4(K(165))^-$  is 4

Final spanned rank in  $S_4(K(165))^+$  is 55

Final spanned rank in  $S_4(K(165))^-$  is 4

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$S_2(K(165))$  is determined by Jacobi restriction and the  $H4Ndd(2,+)$  test  
( $H_4(165,2,2)^+ = 0$ )

So  $S_2(K(165)) = \text{Grit}(J_{\{2,165\}}^{\{\text{cusp}\}})$  (dimension 2)