HOW TO READ TABLES OF FOURIER COEFFICIENTS

9:48, 1–2–2009

Abstract. We explain the format of the Tables of FCs for nonlifts in $S_2^2(K(p))^{\pm}$.

§0. Notation.

$$\mathcal{X}_2 = \{\frac{1}{2}\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \mathcal{P}_2(\mathbb{Q}) : a, b, c \in \mathbb{Z} \text{ and } \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \text{ even}\}$$

$${}^N\mathcal{X}_2 = \{\frac{1}{2}\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \mathcal{X}_2 : 2N|a\}$$

For $T = \frac{1}{2}\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in {}^N\mathcal{X}_2$, we set $\mathrm{Twin}(T) = \frac{1}{2}\left(\begin{smallmatrix} Nc & -b \\ -b & a/N \end{smallmatrix}\right) \in {}^N\mathcal{X}_2$.

$$\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : N|c\}$$

$$\hat{\Gamma}_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) : N|c\}$$

$$\mathbb{P}(\mathbb{Z}/N\mathbb{Z}) = \{\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2 : \gcd(a, b) = 1\}/\sim$$

where $\begin{pmatrix} a \\ b \end{pmatrix} \sim \begin{pmatrix} c \\ d \end{pmatrix}$ if and only if $\exists \mu \in \mathbb{Z} : \mu a \equiv c \mod N$ and $\mu b \equiv d \mod N$.

§1. FCs.

An $f \in S_2^k(K(N))$ has a Fourier expansion

$$f(\Omega) = \sum_{T \in {}^N\mathcal{X}_2} a(T)\, e^{2\pi i \operatorname{tr}(\Omega T)}.$$

For $T = \frac{1}{2}\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, the column "Unreduced form" gives a, b, c. The column "Coeff" gives the value of $a\left(\frac{1}{2}\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\right)$. The "Det" column gives $\det(2T)$. For example, for $p = 277$, the ninth entry row gives

$$a\left(\tfrac{1}{2}\left(\begin{smallmatrix} 277554 & 1825 \\ 1825 & 12 \end{smallmatrix}\right)\right) = -5 \text{ and } \det\left(\begin{smallmatrix} 277554 & 1825 \\ 1825 & 12 \end{smallmatrix}\right) = 23.$$

Note that $277554 = 1002 \cdot 277$ and so

$$\tfrac{1}{2}\left(\begin{smallmatrix} 277554 & 1825 \\ 1825 & 12 \end{smallmatrix}\right) \in {}^{277}\mathcal{X}_2.$$

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Now you can read the FCs from the Table.

Here are a few more elementary comments. For $U \in \hat{\Gamma}_0(N)$, the FCs satisfy

$$a(U'TU) = \det(U)^k \, a(T).$$

Since $k = 2$ in these tables, the FCs are $\hat{\Gamma}_0(N)$-invariant. The "Unreduced form" gives one representative from each $\hat{\Gamma}_0(N)$-class in ${}^N\mathcal{X}_2$. The entries are in order of increasing determinant so that one can see when the $\hat{\Gamma}_0(N)$-classes from ${}^N\mathcal{X}_2$ for a particular determinant are exhausted. Since $D = \det(2T) = ac - b^2 \equiv -b^2 \mod N$, only determinants D with $-D$ a quadratic residue modulo N appear.

§2. Plus and Minus Spaces.

Each eigenform satisfies $f|\mu = \epsilon f$ with $\epsilon = \pm 1$. The FCs then satisfy $a(\mathrm{Twin}(T)) = \epsilon a(T)$. For example, for $D = 23$, we have seen

$$a\left(\frac{1}{2}\begin{pmatrix} 277 \cdot 1002 & 1825 \\ 1825 & 12 \end{pmatrix}\right) = -5.$$

Since our nonlift for $p = 277$ is in the μ-plus space we also have

$$a\left(\frac{1}{2}\begin{pmatrix} 277 \cdot 12 & -1825 \\ -1825 & 1002 \end{pmatrix}\right) = -5.$$

However, since the entries in the "Unreduced form" column are noncanonical, it is not immediately clear which entry this corresponds to.

To remedy this we let $\mathrm{GL}_2(\mathbb{Z})$ act on $\mathcal{X}_2 \times \mathbb{P}(\mathbb{Z}/N\mathbb{Z})$ as $(T, v) \mapsto (U'TU, U^{-1}v)$. If we identify ${}^N\mathcal{X}_2$ with ${}^N\mathcal{X}_2 \times \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\} \subseteq \mathcal{X}_2 \times \mathbb{P}(\mathbb{Z}/N\mathbb{Z})$, then for $T_1, T_2 \in {}^N\mathcal{X}_2$, we have that $(T_1, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to $(T_2, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ if and only if T_1 is $\hat{\Gamma}_0(N)$-equivalent to T_2. For each $T_1 \in {}^N\mathcal{X}_2$, we $\mathrm{GL}_2(\mathbb{Z})$ reduce $(T_1, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ to a (T, v) with T Legendre reduced, that is $0 \leq 2b \leq a \leq c$. The column "Reduced form" lists $a, b, c \quad \alpha, \beta$ for $(T, v) = \left(\frac{1}{2}\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$. This is canonical except for the small number of choices for v: We have the $\mathrm{GL}_2(\mathbb{Z})$-equivalence $(T, v_1) \sim (T, v_2) \iff \exists U \in \mathrm{Aut}_{\mathbb{Z}}(T) : v_1 = Uv_2$.

Going back to the example, let's reduce

$$\left(\tfrac{1}{2}\begin{pmatrix} 277 \cdot 12 & -1825 \\ -1825 & 1002 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \sim \left(\tfrac{1}{2}\begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 27 \end{pmatrix}\right).$$

Using $U = \begin{pmatrix} 11 & -6 \\ 20 & -11 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ we see that

$$U'\begin{pmatrix} 277 \cdot 12 & -1825 \\ -1825 & 1002 \end{pmatrix}U = \begin{pmatrix} 11 & 20 \\ -6 & -11 \end{pmatrix}\begin{pmatrix} 277 \cdot 12 & -1825 \\ -1825 & 1002 \end{pmatrix}\begin{pmatrix} 11 & -6 \\ 20 & -11 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix} \text{ and}$$

$$U^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \left(\begin{smallmatrix}11 & -6\\ 20 & -11\end{smallmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}11\\20\end{pmatrix} \equiv \begin{pmatrix}1\\27\end{pmatrix} \text{ in } \mathbb{P}(\mathbb{F}_{277})$$

since $\begin{vmatrix}11 & 1\\ 20 & 27\end{vmatrix} = 277$. This explains why the FC for the entry $D = 23$ and Reduced form "$4, 1, 6 \quad 1, 27$" is also -5.

If one wishes to move from a reduced form (T, v) to an unreduced form, one simply takes $U'TU$ for a $U \in \mathrm{GL}_2(\mathbb{Z})$ whose first column is $v \mod N$. For example, for $D = 23$ again, take the reduced form from row eleven:

$$(T, v) = \left(\tfrac{1}{2}\left(\begin{smallmatrix}4 & 1\\ 1 & 6\end{smallmatrix}\right), \begin{pmatrix}1\\65\end{pmatrix}\right).$$

For $U = \left(\begin{smallmatrix}1 & 0\\ 65 & 1\end{smallmatrix}\right)$ we have

$$U'TU = \left(\begin{smallmatrix}1 & 65\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}4 & 1\\ 1 & 6\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ 65 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}25484 & 391\\ 391 & 6\end{smallmatrix}\right)$$

which is exactly the unreduced entry because the "Unreduced form" column was constructed in just this way.

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