Paramodularity

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A Survey of the Paramodular Conjecture

1. Part *I*. All elliptic curves defined over $\mathbb{Q}$ are modular.
2. Part *II*. All abelian surfaces defined over $\mathbb{Q}$ with minimal endomorphisms are paramodular.
3. Part *III*. Systematic Evidence for the Paramodular Conjecture
4. Part *IV*. Paramodular Conjecture 2.0
5. Part *V*. Heuristic tables of paramodular forms.

All elliptic curves $E/\mathbb{Q}$ are modular

Theorem (Wiles; Wiles & Taylor; Breuil, Conrad, Diamond & Taylor)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of elliptic curves $E/\mathbb{Q}$ with conductor N*
2. *normalized Hecke eigenforms $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues.*

In this correspondence we have $L(E, s, \mathrm{Hasse}) = L(f, s, \mathrm{Hecke})$.

- Eichler (1954) proved the first examples $L(X_0(11), s, \mathrm{Hasse}) = L(\eta(\tau)^2\eta(11\tau)^2, s, \mathrm{Hecke})$.
- Shimura gave a construction from 2 to 1.
- Weil added $N = N$.

L-functions of elliptic curves over $\mathbb{Q}$

1. The local Hasse p-Euler factor is the characteristic polynomial of Frobenius on the Tate module $\mathbb{T}_\ell(E)$ of the elliptic curve E.

$$Q_p(E,t) = \det\left(I - t\,\mathrm{Frob}_p\,|\mathbb{T}_\ell(E)^{I_p}\right)$$

2. The local Hasse Zeta function can be computed by counting points on the elliptic curve E over finite fields.

$$Z_p(E,t) = \exp\left(\sum_{n=1}^{\infty} \#\{\text{Points on } E/\mathbb{F}_{p^n}\}\frac{t^n}{n}\right) = \frac{Q_p(E,t)}{(1-t)(1-pt)}$$

3. Global Hasse L-function

$$L(E,s,\text{Hasse}) = \prod_{\text{primes } p} Q_p(E,p^{-s})^{-1}$$

An example: LMFDB 11.a3

1. $E[\mathbb{F}] = \{(x, y, z) \in \mathbb{P}^2(\mathbb{F}) : y^2z + yz^2 = x^3 - x^2z\}$

n	1	2	3	4	5	6	7
$\#E[\mathbb{F}_{2^n}]$	5	5	5	25	25	65	145

2. The Hasse Zeta function at $p = 2$:

$$Z_2(E, t) = \exp\left(5t + 5\frac{t^2}{2} + 5\frac{t^3}{3} + 25\frac{t^4}{4} + \cdots\right) = \frac{1 + 2t + 2t^2}{(1 - t)(1 - 2t)}$$

3. $\therefore \quad Q_2(E, t) = 1 + 2t + 2t^2$

An example: LMFDB 11.a3

1. Global Hasse L-function

$$L(E, s, \text{Hasse}) = \prod_{\text{primes } p} Q_p(E, p^{-s})^{-1} = 1 - \frac{2}{2^s} - \frac{1}{3^s} + \frac{2}{4^s} + \frac{1}{5^s} + \cdots$$

2. There is an elliptic modular newform $f_{11} \in S_2(\Gamma_0(11))$

$$\begin{aligned} f_{11}(\tau) &= q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots \\ &= \eta(\tau)^2\eta(11\tau)^2 = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \\ &= \frac{1}{2}\theta\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 8 & 4 \\ 1 & 1 & 4 & 8 \end{pmatrix}(\tau) - \frac{1}{2}\theta\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 6 & 0 \\ 0 & 1 & 0 & 6 \end{pmatrix}(\tau) \end{aligned}$$

All elliptic curves $E/\mathbb{Q}$ are modular (again)

Theorem (Wiles; Wiles & Taylor; Breuil, Conrad, Diamond & Taylor)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of elliptic curves $E/\mathbb{Q}$ with conductor N*
2. *normalized Hecke eigenforms $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues.*

In this correspondence we have $L(E, s, \mathrm{Hasse}) = L(f, s, \mathrm{Hecke})$.

- Credit Taniyama (1956) and Shimura ($\sim$ 1963) for important modularity conjectures.
- In its final form, this is a *classification* theorem.
- Cremona has led the classification of $E/\mathbb{Q}$ up to conductor $N \leq 400\,000$. (johncremona.github.io/ecdtata)

1. What are abelian surfaces $A/\mathbb{Q}$?
2. What are paramodular forms $f \in S_k(K(N))$?
3. How does the Paramodular Conjecture relate the two?

Abelian Varieties

Let $K \subseteq \mathbb{C}$ be an algebraic number field.

Definition

An abelian variety A/K is a projective variety defined over K with an algebraic group law also defined over K.

g=1 Elliptic curves: $A_{\mathrm{hol}} \cong \mathbb{C}^1/\left(\mathbb{Z} + \tau\mathbb{Z}\right)$ for $\tau \in \mathcal{H}$.

g=2 Abelian surfaces: $A_{\mathrm{hol}} \cong \mathbb{C}^2/\left(D\mathbb{Z}^2 + Z\mathbb{Z}^2\right)$ for $Z \in \mathcal{H}_2$.

- $D = \mathrm{diag}(1, d)$ for $d \in \mathbb{N}$ gives the type of "polarization" of A.
- $\mathcal{H}_2 = \{Z \in M_{2\times 2}^{\mathrm{sym}}(\mathbb{C}) : \mathrm{Im}\, Z > 0\}$, the *Siegel upper half space.*
- But for $g > 1$ the equations defining A inside a projective space are *complicated!*

How can we construct *some* abelian surfaces?

1. If $\mathcal{C}$ is a curve of genus 2 defined over $\mathbb{Q}$, then

$$A = \mathrm{Jac}(\mathcal{C})$$

   is an abelian surface defined over $\mathbb{Q}$ with *principal* polarization, i.e., type $D = \mathrm{diag}(1, 1)$.

2. If a genus 3 curve $\mathcal{C}_3$ is a ramified double cover of a genus 1 curve $\mathcal{C}_1$, then the abelian surface

$$A = \mathrm{Prym}(\mathcal{C}_3/\mathcal{C}_1) = \mathrm{Jac}(\mathcal{C}_3)/\,\mathrm{Jac}(\mathcal{C}_1)$$

   has a natural polarization of type $D = \mathrm{diag}(1, 2)$.

Definition of Siegel Modular Forms

- Siegel Upper Half Space: $\mathcal{H}_n = \{Z \in M^{\mathrm{sym}}_{n\times n}(\mathbb{C}) : \operatorname{Im} Z > 0\}$.
- Symplectic group: $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ acts on $Z \in \mathcal{H}_n$ by $\sigma \cdot Z = (AZ + B)(CZ + D)^{-1}$.
- $\Gamma \subseteq \mathrm{Sp}_n(\mathbb{R})$ such that $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ has finite index in Γ and $\mathrm{Sp}_n(\mathbb{Z})$
- Slash action: For $f : \mathcal{H}_n \to \mathbb{C}$ and $\sigma \in \mathrm{Sp}_n(\mathbb{R})$, $(f|_k\sigma)(Z) = \det(CZ + D)^{-k} f(\sigma \cdot Z)$.
- Siegel Modular Forms: $M_k(\Gamma)$ is the $\mathbb{C}$-vector space of holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ that are "bounded at the cusps" and that satisfy $f|_k\sigma = f$ for all $\sigma \in \Gamma$.
- Cusp Forms: $S_k(\Gamma) = \{f \in M_k(\Gamma)$ that "vanish at the cusps"$\}$

Definition of paramodular form

- A *paramodular form* is a Siegel modular form for a paramodular group. In degree 2, the paramodular group of level N, is

$$\Gamma = K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

- $K(N)$ is the stabilizer in $\mathrm{Sp}_2(\mathbb{Q})$ of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$.
- ${}^{T}K(N)\backslash\mathcal{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$. (T is "transpose" here.)
- The paramodular Fricke involution splits paramodular forms into plus and minus spaces.

$$S_k\left(K(N)\right) = S_k\left(K(N)\right)^{+} \oplus S_k\left(K(N)\right)^{-}$$

All abelian surfaces $A/\mathbb{Q}$ with a minimal endomorphism group over $\mathbb{Q}$ are paramodular

Paramodular Conjecture (Brumer and Kramer 2009)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of abelian surfaces $A/\mathbb{Q}$ with conductor N and endomorphisms $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$,*
2. *lines of Hecke eigenforms $f \in S_2(K(N))^{\mathrm{new}}$ that have rational eigenvalues and are not Gritsenko lifts from $J_{2,N}^{\mathrm{cusp}}$.*

In this correspondence we have

$$L(A, s, \text{Hasse-Weil}) = L(f, s, \text{spin}).$$

- The direction $1 \to 2$ is still believed, but $2 \to 1$ will be amended later.

Remarks

- The endomorphisms *defined over* $\mathbb{Q}$ should be minimal: $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. (Where is this condition in the elliptic case?) The Paramodular conjecture addresses the typical case. When $\mathrm{End}_{\mathbb{Q}}(A) > \mathbb{Z}$, modularity is known.
- Newform theory for paramodular groups: Ibukiyama 1984; Roberts and Schmidt 2007 (LNM 1918).
- Grit : $J_{k,N}^{\mathrm{cusp}} \to S_k\,(K(N))$, the Gritsenko lift from Jacobi cusp forms of index N to paramodular cusp forms of level N is an advanced version of the Maass lift.

Do the arithmetic and automorphic data match up?

Looks like it.

By 1997, Brumer has a long list of N that could possibly be the conductor of an abelian surface $A/\mathbb{Q}$ and begins campaigning for the computation of corresponding automorphic forms.

Theorem (PY 2009)

Let $p < 600$ be prime. If $p \notin \{277, 349, 353, 389, 461, 523, 587\}$ then $S_2(K(p))$ consists entirely of Gritsenko lifts.

This exactly matches Brumer's "Yes" list for prime levels $p < 600$.

L-functions of abelian surfaces over $\mathbb{Q}$

1. The local Hasse-Weil p-Euler factor is the characteristic polynomial of Frobenius on the Tate module $\mathbb{T}_\ell(A)$ of the abelian surface A.

$$Q_p(A,t) = \det\left(I - t\,\mathrm{Frob}_p\,|\mathbb{T}_\ell(A)^{I_p}\right)$$

2. When the isogeny class of $A/\mathbb{Q}$ contains a Jacobian or a Prym then computation of the Hasse-Weil p-Euler factors will be easy. However, it may be difficult to find a representative abelian surface.

3. Global Hasse-Weil L-function

$$L(A,s,\text{H-W}) = \prod_{\text{primes } p} Q_p(A,p^{-s})^{-1}$$

L-functions of abelian surfaces over $\mathbb{Q}$

1. For $N < 1000$, we have seen mainly Jacobians, Pryms, and Weil restrictions, but the algorithmic challenges searching for $A/\mathbb{Q}$ will grow with the conductor N.
2. In the special case when $A = \mathrm{Jac}(C)$ is the Jacobian of a genus two curve C, we have

$$L(A, s, \text{H-W}) = L(C, s, \text{H-W})$$

3. The local Hasse-Weil p-Euler factors for $A = \mathrm{Jac}(C)$ are accessible by counting points

$$Z_p(C, t) = \exp\left(\sum_{n=1}^{\infty} \#\{\text{Points on } C/\mathbb{F}_{p^n}\} \frac{t^n}{n}\right) = \frac{Q_p(C, t)}{(1-t)(1-pt)}$$

An example: LMFDB: Genus 2 Curve 277.*a*.277.1

1. $A_{277} = \mathrm{Jac}(C_{277})$ for C_{277} given by the hyperelliptic curve $y^2 + (x^3 + x^2 + x + 1)y = -x^2 - x$
2. Magma will compute Hasse-Weil Euler factors of a curve:

```
> G:=x^3 + x^2 + x + 1; F:=-x^2 - x;
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($y^2 + G(x)y = F(x)$ is the equation)

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> C:=HyperellipticCurve(F,G);
> J:=Jacobian(C);
> h2:=EulerFactor(J,GF(2)); h2;
4*x^4 + 4*x^3 + 4*x^2 + 2*x + 1
> h3:=EulerFactor(J,GF(3)); h3;
9*x^4 + 3*x^3 + x^2 + x + 1
> h5:=EulerFactor(J,GF(5)); h5;
25*x^4 + 5*x^3 - 2*x^2 + x + 1
```

An example: a paramodular newform of level 277

- How can we make a newform in $S_2(K(277))$?
- Both *Gritsenko lifts* and *Borcherds products* can help.
  Both are built from *Jacobi forms.*
- Dimension formulas. No— not in weight two.
- Theta series. No— the trace from $\Gamma_0(p) \to K(p)$ gives zero.
- Modular symbols. — no such theory yet.

Fourier-Jacobi expansions

- Fourier expansion of Siegel modular form:

$$f(Z) = \sum_{T \geq 0} a(T; f) e(\mathrm{tr}(ZT))$$

- Fourier expansion of paramodular form $f \in M_k(K(N))$ in coordinates:

$$f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{\substack{n,r,m \in \mathbb{Z}: \\ n,m \geq 0,\, 4Nnm \geq r^2}} a\left(\begin{pmatrix} n & r/2 \\ r/2 & Nm \end{pmatrix}; f\right) e(n\tau + rz + Nm\omega)$$

- Fourier-Jacobi expansion of paramodular form $f \in M_k(K(N))$:

$$f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m \in \mathbb{Z}:\, m \geq 0} \phi_m(\tau, z) e(Nm\omega)$$

Fourier-Jacobi expansion (FJE)

$$\text{FJE: } f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m \in \mathbb{Z}:\, m \geq 0} \phi_m(\tau, z) e(Nm\omega)$$

The Fourier-Jacobi expansion of a paramodular form is fixed *term-by-term* by the following subgroup of the paramodular group $K(N)$:

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

- $P_{2,1}(\mathbb{Z})/\{\pm I\} \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathrm{Heisenberg}(\mathbb{Z})$

Thus the coefficients ϕ_m are automorphic forms in their own right and easier to compute than Siegel modular forms. This is one motivation for the introduction of Jacobi forms.

Definition of Jacobi Forms: Automorphicity

Level one

- Assume $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is holomorphic.

$$\begin{aligned} \tilde{\phi} &: \mathcal{H}_2 \to \mathbb{C} \\ \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) &\mapsto \phi(\tau, z) e(m\omega) \end{aligned}$$

- Assume that $\tilde{\phi}$ transforms by $\chi \det(CZ + D)^k$ for

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

Definition of Jacobi Forms: Support

- Jacobi forms are tagged with additional adjectives to reflect the support $\mathrm{supp}(\phi) = \{(n, r) \in \mathbb{Q}^2 : c(n, r; \phi) \neq 0\}$ of the Fourier expansion

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(n, r; \phi) q^n \zeta^r, \qquad q = e(\tau), \zeta = e(z).$$

- $\phi \in J_{k,m}^{\mathrm{cusp}}$: automorphic and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 > 0$
- $\phi \in J_{k,m}$: automorphic and $c(n, r; \phi) \neq 0 \implies 4mn - r^2 \geq 0$
- $\phi \in J_{k,m}^{\mathrm{weak}}$: automorphic and $c(n, r; \phi) \neq 0 \implies n \geq 0$
- $\phi \in J_{k,m}^{\mathrm{wh}}$: automorphic and $c(n, r; \phi) \neq 0 \implies n >> -\infty$ ("wh" stands for *weakly holomorphic*)

Index Raising Operators $V(\ell) : J_{k,m} \to J_{k,m\ell}$

from Eichler-Zagier

The Jacobi $V(\ell)$ are images of the elliptic $T(\ell)$.

Elliptic Hecke Algebra $\longrightarrow$ Jacobi Hecke Algebra

$$\sum \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \sum P_{2,1}(\mathbb{Z}) \begin{pmatrix} a & 0 & b & 0 \\ 0 & ad-bc & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sum_{\substack{ad=\ell \\ b \bmod d}} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \sum_{\substack{ad=\ell \\ b \bmod d}} P_{2,1}(\mathbb{Z}) \begin{pmatrix} a & 0 & b & 0 \\ 0 & \ell & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T(\ell) \mapsto V(\ell)$$

Fourier-Jacobi expansion of the Gritsenko lift

Any Jacobi cusp form can be the leading Fourier-Jacobi coefficient of a paramodular form.

Theorem (Gritsenko)

For $\phi \in J_{k,m}^{\mathrm{cusp}}$ *the series* $\mathrm{Grit}(\phi)$ *converges and defines a map*

$$\mathrm{Grit} : J_{k,m}^{\mathrm{cusp}} \to S_k\left(K(m)\right)^{\epsilon}, \quad \epsilon = (-1)^k.$$

$$\mathrm{Grit}(\phi)\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{\ell \in \mathbb{N}} (\phi | V_\ell)(\tau, z) e(\ell m \omega).$$

Borchers Product Summary

Theorem (Borcherds, Gritsenko, Nikulin)

Given $\psi \in J^{\mathrm{wh}}_{0,N}(\mathbb{Z})$*, a weakly holomorphic weight zero, index* N *Jacobi form with integral coefficients*

$$\psi(\tau, z) = \sum_{n,r \in \mathbb{Z}:\, n \geq -N_o} c\,(n, r)\, q^n \zeta^r$$

there is a weight $k' \in \mathbb{Z}$*, a character* χ*, and a* meromorphic *paramodular form* $\mathrm{Borch}(\psi) \in M^{\mathrm{mero}}_{k'}(K(N))(\chi)$

$$\mathrm{Borch}(\psi)(Z) = q^A \zeta^B \xi^C \prod_{n,m,r \in \mathbb{Z}} \left(1 - q^n \zeta^r \xi^{Nm}\right)^{c(nm,r)}$$

converging in a nghd of infinity and defined by analytic continuation.

Theta Blocks: a great way to make Jacobi forms

due to Gritsenko, Skoruppa, and Zagier

- Dedekind Eta function $\eta \in J^{\mathrm{cusp}}_{1/2,0}(\epsilon)$

$$\eta(\tau) = q^{1/24} \prod_{n\in\mathbb{N}} (1 - q^n)$$

- Odd Jacobi Theta function $\vartheta \in J^{\mathrm{cusp}}_{1/2,1/2}(\epsilon^3 v_H)$

$$\vartheta(\tau, z) = q^{1/8} \left(\zeta^{1/2} - \zeta^{-1/2}\right) \prod_{n\in\mathbb{N}} (1 - q^n)(1 - q^n\zeta)(1 - q^n\zeta^{-1})$$

- $\mathrm{TB}_k[d_1, d_2, \ldots, d_\ell](\tau, z) = \eta(\tau)^{2k-\ell} \prod_{j=1}^{\ell} \vartheta(\tau, d_j z) \in J^{\mathrm{mero}}_{k,m}(\epsilon^{2k+2\ell})$ where $2m = d_1^2 + d_2^2 + \cdots + d_\ell^2$ and $d_i \in \mathbb{N}$.

There are 10 dimensions of Gritsenko lifts in $S_2(K(277))$

We have $\dim S_2(K(277)) = 11$ whereas the dimension of Gritsenko lifts in $S_2(K(277))$ is $\dim J_{2,277}^{\mathrm{cusp}} = 10$.

Let $G_i = \mathrm{Grit}(\mathrm{TB}_2(\Sigma_i)) \in S_2(K(277))$ for $1 \le i \le 10$ be the lifts of the 10 theta blocks given by:

$\Sigma_i \in \{\, [2, 4, 4, 4, 5, 6, 8, 9, 10, 14], [2, 3, 4, 5, 5, 7, 7, 9, 10, 14],$
$[2, 3, 4, 4, 5, 7, 8, 9, 11, 13], [2, 3, 3, 5, 6, 6, 8, 9, 11, 13],$
$[2, 3, 3, 5, 5, 8, 8, 8, 11, 13], [2, 3, 3, 5, 5, 7, 8, 10, 10, 13],$
$[2, 3, 3, 4, 5, 6, 7, 9, 10, 15], [2, 2, 4, 5, 6, 7, 7, 9, 11, 13],$
$[2, 2, 4, 4, 6, 7, 8, 10, 11, 12], [\, 2, 2, 3, 5, 6, 7, 9, 9, 11, 12] \,\}.$

Remark: The Gritsenko lifts of these ten theta blocks are all Borcherds products as well.

The nonlift paramodular eigenform $f_{277} \in S_2(K(277))$

$$f_{277} = \frac{Q}{L} \qquad \text{(proven holomorphic)}$$

$$\begin{aligned}
Q = {} & -14G_1^2 - 20G_8G_2 + 11G_9G_2 + 6G_2^2 - 30G_7G_{10} + 15G_9G_{10} + 15G_{10}G_1 \\
& - 30G_{10}G_2 - 30G_{10}G_3 + 5G_4G_5 + 6G_4G_6 + 17G_4G_7 - 3G_4G_8 - 5G_4G_9 \\
& - 5G_5G_6 + 20G_5G_7 - 5G_5G_8 - 10G_5G_9 - 3G_6^2 + 13G_6G_7 + 3G_6G_8 \\
& - 10G_6G_9 - 22G_7^2 + G_7G_8 + 15G_7G_9 + 6G_8^2 - 4G_8G_9 - 2G_9^2 + 20G_1G_2 \\
& - 28G_3G_2 + 23G_4G_2 + 7G_6G_2 - 31G_7G_2 + 15G_5G_2 + 45G_1G_3 - 10G_1G_5 \\
& - 2G_1G_4 - 13G_1G_6 - 7G_1G_8 + 39G_1G_7 - 16G_1G_9 - 34G_3^2 + 8G_3G_4 \\
& + 20G_3G_5 + 22G_3G_6 + 10G_3G_8 + 21G_3G_9 - 56G_3G_7 - 3G_4^2, \\
L = {} & -G_4 + G_6 + 2G_7 + G_8 - G_9 + 2G_3 - 3G_2 - G_1.
\end{aligned}$$

A newform $f_{587}^{-} \in S_2(K(587))^{-}$.

Gritsenko, P–, Yuen (2016)

Construct theta blocks $\phi \in J_{2,587}^{\mathrm{cusp}}$ and $\Xi \in J_{2,2\cdot 587}^{\mathrm{cusp}}$:

$$\phi = \mathrm{TB}_2[1,1,2,2,2,3,3,4,4,5,5,6,6,7,8,8,9,10,11,12,13,14]$$
$$\Xi = \mathrm{TB}_2[1,10,2,2,18,3,3,4,4,15,5,6,6,7,8,16,9,10,22,12,13,14]$$

$$\psi = \frac{\phi\,|V(2) - \Xi}{\phi} \in J_{0,587}^{\mathrm{wh}}(\mathbb{Z})$$

$$\psi(\tau, z) = \sum_{n,r} c(n,r) q^n \zeta^r = 4 + \frac{1}{q} + \zeta^{-14} + \cdots + q^{134}\zeta^{561} + \cdots$$

$$f_{587}^{-}\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = q^2 \zeta^{68} \xi^{587} \prod_{(n,m,r)\geq 0} \left(1 - q^n \zeta^r \xi^{Nm}\right)^{c(nm,r)}$$

A nonlift newform $f_{249} \in S_2(K(249))$.

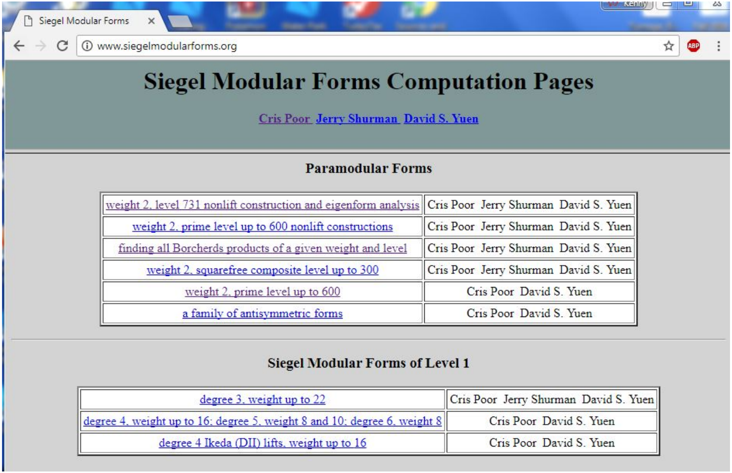
P–, Shurman, Yuen (2017)

- $\psi_{249}(\tau,z) = \dfrac{\vartheta(\tau,8z)}{\vartheta(\tau,z)} \dfrac{\vartheta(\tau,18z)}{\vartheta(\tau,6z)} \dfrac{\vartheta(\tau,14z)}{\vartheta(\tau,7z)} \in J^{\mathrm{w.h.}}_{0,249}(\mathbb{Z})$

$$f_{249}\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = 14\, q^2 \zeta^{63} \xi^{498} \prod_{\substack{n,m,r \in \mathbb{Z}:\ m \geq 0 \\ \text{if } m=0 \text{ then } n \geq 0 \\ \text{if } m=n=0 \text{ then } r<0}} (1 - q^n \zeta^r \xi^{mN})^{c(nm,r;\,\psi_{249})}$$
$$- 6\ \mathrm{Grit}(\mathrm{TB}_2(2,3,3,4,5,6,7,9,10,13))$$
$$- 3\ \mathrm{Grit}(\mathrm{TB}_2(2,2,3,5,5,6,7,9,11,12))$$
$$+ 3\ \mathrm{Grit}(\mathrm{TB}_2(1,3,3,5,6,6,6,9,11,12))$$
$$+ 2\ \mathrm{Grit}(\mathrm{TB}_2(1,1,2,3,4,5,6,9,10,15))$$
$$+ 7\ \mathrm{Grit}(\mathrm{TB}_2(1,2,3,3,4,5,6,9,11,14)).$$

siegelmodularforms.org

- Our paramodular website: www.siegelmodularforms.org
- (Joint with J. Shurman, D. Yuen.)



Siegel Modular Forms Computation Pages

Cris Poor Jerry Shurman David S. Yuen

Paramodular Forms

weight 2, level 731 nonlift construction and eigenform analysis	Cris Poor Jerry Shurman David S. Yuen
weight 2, prime level up to 600 nonlift constructions	Cris Poor Jerry Shurman David S. Yuen
finding all Borcherds products of a given weight and level	Cris Poor Jerry Shurman David S. Yuen
weight 2, squarefree composite level up to 300	Cris Poor Jerry Shurman David S. Yuen
weight 2, prime level up to 600	Cris Poor David S. Yuen
a family of antisymmetric forms	Cris Poor David S. Yuen

Siegel Modular Forms of Level 1

degree 3, weight up to 22	Cris Poor Jerry Shurman David S. Yuen
degree 4, weight up to 16; degree 5, weight 8 and 10; degree 6, weight 8	Cris Poor David S. Yuen
degree 4 Ikeda (DII) lifts, weight up to 16	Cris Poor David S. Yuen

Pulling back: G a reductive algebraic group

Modularity

Arithmetic | Automorphic

Motives $\longrightarrow$ Galois representations $\longleftarrow$ Automorphic reps

Motives $\longrightarrow \rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \hat{G}(\mathbb{C}) \longleftarrow$ Auto reps of $G(\mathbb{A})$

$\uparrow$ $\uparrow$

Etale cohomology of varieties | Automorphic forms

Abelian surfaces $\to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(4, \mathbb{C}) \leftarrow$ paramodular, $G = SO(5)$

Abelian surfaces $\to$ L-functions $\leftarrow$ paramodular f, $G = SO(5)$

Elliptic curves $\to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2, \mathbb{C}) \leftarrow$ elliptic $G = \mathrm{GL}(2)$

Proven paramodularity!

- Q: How can you prove the L-functions of A and f agree?
- A: You associate Galois representations to each of A and f and prove that their Galois representations are equivalent.
- Q: How can you prove two Galois representations are equivalent?
- A: Use a generalization of Faltings-Serre to GSp(4) to show that the Galois representations agree if enough traces of Frobenius agree.
- The Hasse-Weil and spin L-functions of some A and f agree.
  $N = 277$: need primes up to $p = 43$
  $N = 353$: need primes as large as $p = 137$
  $N = 587^-$: need primes up to $p = 41$
  (On arXiv— *On the paramodularity of typical abelian surfaces,* Brumer, Pacetti, Poor, Tornarìa, Voight, Yuen)

Modularity of A_{731}, a composite level: $731 = 17 \cdot 43$

1. Berger and Klosin have proven the modularity of

$$A_{731} = \mathrm{Jac}\left(y^2 = x^6 - 6x^4 + 4x^3 + 9x^2 - 16x - 4\right),$$

   the first modularity proof of a *typical* abelian surface with composite conductor.

2. A_{731} has rational 5-torsion.

   The candidate eigenform $f_{731} \in S_2\left(K(731)\right)$ is congruent to a Gritsenko lift modulo $p = 5$.

3. The Galois representations σ_A and σ_f associated to A_{731} and f_{731} are the unique characteristic zero deformation of $\bar{\sigma}_A$.

Tobias Berger and Krzysztof Klosin: *Deformations of Saito-Kurokawa type and the Paramodular Conjecture.*

arXiv:1710.10228v2 (appendix by Poor, Shurman, Yuen)

Part Three

Systematic Evidence for the Paramodular Conjecture

Twisting

Let χ be the nontrivial quadratic Dirichlet character modulo p.

- Twist L-functions: $\sum_n \frac{a_n}{n^s} \mapsto \sum_n \chi(n) \frac{a_n}{n^s}$
- Twist abelian surfaces: $A \mapsto A^\chi$
  I only indicate the special case of Jacobians:
  $\mathrm{Jac}\left(y^2 = f(x)\right) \mapsto \mathrm{Jac}\left(py^2 = f(x)\right)$.
- According to the Paramodular Conjecture, there should be a way to twist paramodular forms. (Well, weight two newforms . . .)

$$\begin{array}{ccc} A & \xrightarrow{\text{twist}} & A^\chi \\ {\scriptstyle \mathrm{PC}}\downarrow & & \downarrow{\scriptstyle \mathrm{PC}} \\ f & \xrightarrow{??} & f^\chi \end{array}$$

Twisting

Johnson-Leung and Roberts have a theory of twisting paramodular forms.

- Jennifer Johnson-Leung and Brooks Roberts: *Twisting of Siegel Paramodular Forms,* Int. J. Number Theory 13 (2017).
- **Theorem.** Let χ be the nontrivial quadratic Dirichlet character modulo an odd prime $p \nmid N$. There exists a linear twisting map

$$\mathcal{T}_\chi : S_k\left(K(N)\right) \to S_k\left(K(Np^4)\right)$$

  such that if f is a new eigenform and $\mathcal{T}_\chi(f) \neq 0$ then

$$L\left(\mathcal{T}_\chi(f), s, \mathrm{spin}\right) = L^\chi\left(f, s, \mathrm{spin}\right).$$

- Therefore if f shows the modularity of A then $\mathcal{T}_\chi(f)$ shows the modularity of A^χ.
- For example, for all $p \neq 277$, the surface A_{277}^χ is modular!

Weil Restriction

Let E/K be an elliptic curve over a quadratic field K.

There exists an abelian surface over $\mathbb{Q}$, $A_E = \mathrm{Res}_{K/\mathbb{Q}}(E)$, the *Weil restriction* of E, whose $\mathbb{Q}$-points are in bijection with the K-points of E.

If E is *not* isogenous to its conjugate over K then $\mathrm{End}_{\mathbb{Q}}(A_E) = \mathbb{Z}$, and the Paramodular Conjecture applies to the Weil restriction A_E.

- When K is real quadratic, E/K is modular with respect to some weight $(2, 2)$ Hilbert modular form F for $\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a})$.

  Nuno Freitas and Bao V. Le Hung and Samir Siksek: *Elliptic curves over real quadratic fields are modular.* Invent. Math. 201 (2015)

- According to the Paramodular Conjecture, there should be a lift from Hilbert to paramodular forms.

Weil Restriction

Johnson-Leung and Roberts have a theory of lifting Hilbert eigenforms to paramodular eigenforms

- Jennifer Johnson-Leung, and Brooks Roberts: *Siegel modular forms of degree two attached to Hilbert modular forms.* J. Number Theory 132 (2012)
- For real quadratic K, the modularity of the Weil restrictions $A_E = \mathrm{Res}_{K/\mathbb{Q}}(E)$ is shown by the J-LR paramodular lift of the Hilbert eigenform that shows the modularity of E/K.

$$\begin{array}{ccc} E/K & \xrightarrow{\quad \text{WR} \quad} & A/\mathbb{Q} \\ {\scriptstyle \text{Modularity}}\downarrow & & \downarrow{\scriptstyle \text{PC}} \\ \text{Hilbert form} & \xrightarrow{\text{JL-R lift}} & \text{Paramodular form} \end{array}$$

Weil Restriction

For imaginary quadratic K, Berger, Dembélé, Pacetti, and Sengun have a similar theory lifting Bianchi eigenforms to paramodular eigenforms.

- Tobias Berger, and Lassina Dembélé, and Ariel Pacetti, and Mehmet Haluk Sengun: *Theta lifts of Bianchi modular forms and applications to paramodularity.* J. Lond. Math. Soc. (2) 92 (2015)

$$\begin{array}{ccc} E/K & \xrightarrow{\quad\text{WR}\quad} & A/\mathbb{Q} \\ {\scriptstyle\text{modularity}}\downarrow & & \downarrow{\scriptstyle\text{PC}} \\ \text{Bianchi form} & \xrightarrow{\text{BDPS lift}} & \text{Paramodular form} \end{array}$$

Do the arithmetic and automorphic data match up?

Theorem (P– Yuen 2009)

Let $p < 600$ be prime. If $p \notin \{277, 349, 353, 389, 461, 523, 587\}$ then $S_2(K(p))$ consists entirely of Gritsenko lifts.

Paramodular Conjecture verified for prime levels $p < 600$ not listed above. Brumer and Kramer prove the absence of abelian surfaces.

- The dimension formula of Ibukiyama (2007: Hamana Lake) for $S_4\left(K(p)\right)$ was crucial to these computations.

Do the arithmetic and automorphic data match up?

Theorem (Breeding, P– Yuen 2016)

For all $N \leq 60$, $S_2(K(N))$ consists entirely of Gritsenko lifts.

Paramodular Conjecture verified for many odd levels $N \leq 60$. Brumer and Kramer prove the absence of semistable abelian surfaces of odd conductor.

- We proved an *a priori* bound on the number of Fourier-Jacobi coefficients needed to determine the space $S_2\left(K(N)^*, \chi\right)$.
- Jeffery Breeding, Cris Poor, and David S. Yuen: *Computations of spaces of paramodular forms of general level.* J. Korean Math. Soc. 53 (2016)

Do the arithmetic and automorphic data match up?

Theorem (P– Shurman, Yuen 2017)

Let $N < 300$ be square-free. If $N \notin \{249, 277, 295\}$ then $S_2(K(N))$ consists entirely of Gritsenko lifts. Furthermore, there is exactly one dimension of nonlift eigenforms for $S_2(K(249)), S_2(K(277)), S_2(K(295))$.

Paramodular Conjecture verified for odd squarefree levels $N < 300$, except as noted.

Cris Poor, Jerry Shurman, David S. Yuen: *Siegel paramodular forms of weight 2 and squarefree level,* Int. J. Number Theory 13 (2017)

- The dimension formula of Ibukiyama and Kitayama for $S_4(K(N))$ and squarefree N was crucial to these computations.

Part Four

The Paramodular Conjecture 2.0

Perspective on classification results.

- Elliptic curves over $\mathbb{Q}$ $\leftrightarrow$ elliptic $\mathbb{Q}$-newforms in $S_2(\Gamma_0(N))$
- E over real quad K $\rightarrow$ Hilbert $\mathbb{Q}$-newforms in $S_2(\mathrm{SL}(\mathcal{O}_K \oplus \mathfrak{a}))$ (but ($\leftarrow$) is not quite done— see discussion in Freitas, Siksek: arxiv.org/pdf/1307.3162.pdf)
- E over imaginary quad K $\rightarrow$ Bianchi $\mathbb{Q}$-newforms in $S_2(\Gamma_0(\mathfrak{n}))$

But there is a problem going ($\leftarrow$) from Bianchi newforms to E/K as noted by John Cremona.

Counterexamples

Ciaran Schembri is my source for this example.

- Define a hyperelliptic curve $C_o/\mathbb{Q}(i)$ of genus two by $y^2 = x^6 + 4ix^5 - (6+2i)x^4 + (7-i)x^3 - (9-8i)x^2 - 10ix + (3+4i)$
- $A_o = \mathrm{Jac}(C_o)$ is an abelian surface over $\mathbb{Q}(i)$ of conductor $\mathfrak{p}_{5,1}^4\mathfrak{p}_{37,2}^4$ of norm $34225^2 = 185^4$.
- $\mathcal{O}_6 \hookrightarrow \mathrm{End}_{\mathbb{Q}(i)}(A_o)$ where $\mathcal{O}_6$ is the maximal order of the rational quaternion algebra of discriminant 6
- There is a Bianchi newform $f_o \in S_2(\Gamma_0(\mathfrak{p}_{5,1}^4\mathfrak{p}_{37,2}^4))$ with $\mathbb{Q}$-rational eigenvalues such that $L(A_o, s, \mathrm{Hasse\text{-}Weil}) = L(f_o, s)^2$.
- By Faltings, there can be no $E/\mathbb{Q}(i)$ with $L(E, s, \mathrm{Hasse}) = L(f_o, s)$. ($L$-functions determine isogeny classes of abelian varieties)

Thus, the pairing between $E/\mathbb{Q}(i)$ and Bianchi newforms is not perfect.

A counterexample to the Paramodular conjecture 1.0

Frank Calegari pointed out counterexamples to the Paramodular Conjecture in January, 2018.

- By Weil restriction, $B = \mathrm{WR}(A_o/\mathbb{Q}(i))$ is an abelian fourfold defined over $\mathbb{Q}$ with $\mathrm{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q}$ an indefinite quaternion algebra.
- The lift of Berger, Dembélé, Pacetti, and Sengun gives $f = \text{BDPS-lift}(f_o) \in S_2(K(N))$. Note $N = (16 \cdot 185)^2 = 8\,761\,600$.
- $L(B, s, \mathrm{H\text{-}W}) = L(f, s, \mathrm{spin})^2$ and there can be no abelian surface $A/\mathbb{Q}$ with $L(A, s, \mathrm{H\text{-}W}) = L(f, s, \mathrm{spin})$ due to the different endomorphism rings $\mathrm{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q} \neq \mathrm{End}_{\mathbb{Q}}(A \oplus A) \otimes \mathbb{Q}$, these being isogeny invariants.

The Paramodular conjecture 2.0 (2018)

An abelian fourfold $B/\mathbb{Q}$ has *quaternionic multiplication* (QM) if $\mathrm{End}_{\mathbb{Q}}(B)$ is an order in a non-split quaternion algebra over $\mathbb{Q}$. A cuspidal, nonlift Siegel paramodular newform $f \in S_2(K(N))$ with rational Hecke eigenvalues will be called a *suitable* paramodular form of level N.

Paramodular Conjecture (Brumer–Kramer)

Let $N \in \mathbb{N}$. Let $\mathcal{A}_N$ be the set of isogeny classes of abelian surfaces $A/\mathbb{Q}$ of conductor N with $\mathrm{End}_{\mathbb{Q}} A = \mathbb{Z}$. Let $\mathcal{B}_N$ be the set of isogeny classes of QM abelian fourfolds $B/\mathbb{Q}$ of conductor N^2. Let $\mathcal{P}_N$ be the set of suitable paramodular forms of level N, up to nonzero scaling. There is a bijection $\mathcal{A}_N \cup \mathcal{B}_N \leftrightarrow \mathcal{P}_N$ such that

$$L(C, s, \mathrm{H\text{-}W}) = \begin{cases} L(f, s, \mathrm{spin}), & \text{if } C \in \mathcal{A}_N, \\ L(f, s, \mathrm{spin})^2, & \text{if } C \in \mathcal{B}_N. \end{cases}$$

Brumer and Kramer: QM implies $N = M^2 s$ with $s \mid \gcd(30, M)$.

Part Five

Heuristic Tables for the Paramodular Conjecture

(found by classifying initial Fourier-Jacobi expansions)

With updated dimensions from: *Nonlift weight two paramodular eigenform constructions,* by Poor, Shurman, and Yuen. In progress.

And from: *Antisymmetric paramodular forms of weights* 2 *and* 3, by Gritsenko, Poor, and Yuen: arXiv:1609.04146v1.

Heuristic tables: $k = 2$ paramodular newforms: $N \le 800$.

$$+\text{new nonlift} = \dim\left((S_2(K(N))^{\text{new}})^+ \,/\, \text{Grit}\left(J_{2,N}^{\text{cusp}}\right)\right)$$

$$-\text{new} = \dim\left(S_2(K(N))^{\text{new}}\right)^- .$$

The "=" means "proven."

N	$+$new	$-$new	various comments
249	$= 1$		BP+Grit; Jac
277	$= 1$		modular! Q/L; Jac
295	$= 1$		BP+Grit; Jac
349	$= 1$		BP+Grit; Jac
353	$= 1$		modular! BP+Grit; Jac
388	1		Jac

N	$+$new	$-$new	various comments
389	$=1$		BP+Grit; Jac
394	1		Jac
427	1		Jac
461	$=1$		Tr(BP)+Grit; Jac
464	1		Jac
472	1		Jac
511	2		quad pair, $\sqrt{5}$; 4-dim $A/\mathbb{Q}$?
523	$=1$		BP+Grit; Jac
550	1		no match known

N	$+$new	$-$new	various comments
555	1		Jac
561	1		Prym
574	1		Jac
587	$=1$	$=1$ *modular*!	Tr(BP)+Grit and BP-; Jacs
597	1		Jac
603	1		Jac
604	1		Jac
623	1		Jac
633	1		Jac

N	$+$new	$-$new	various comments
637	2		quad pair, $\sqrt{2}$; 4-dim $A/\mathbb{Q}$?
644	1		Jac
645	2		quad pair, $\sqrt{2}$; 4-dim $A/\mathbb{Q}$?
657	≥ 1		modular! WR: $E_{(9\zeta_6-8)}/\mathbb{Q}(\sqrt{-3})$
665	1		Prym
688	1		Jac
691	1		Jac
702	1		no match known

$$\zeta_6 = \exp(2\pi i \tfrac{1}{6})$$

N	$+$new	$-$new	various comments
704	1		Jac
708	1		Jac
709	1		Jac
713	1	≥ 1	BP-; Jacs
731	$= 1$		Berger and Klosin: modular! Jac
737	1		Prym
741	1		Jac
743		1	Jac

N	$+$new	$-$new	various comments
745	1		Jac
760	1		no match known
762	1		Jac
763	1		Jac
768	1		Jac
775	≥ 1		modular! WR: $E_{(5\phi-2)}/\mathbb{Q}(\sqrt{5})$
797	1		Jac

$$\phi = \frac{1+\sqrt{5}}{2}$$

Thanks to Armand Brumer for all his help, in particular for providing me with the majority of the abelian surfaces in this talk.

Thank you!